Comment on "Multicomponent turbulence, the spherical limit, and non-Kolmogorov spectra"

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It is shown that the generalization of the Navier-Stokes equations to a theory with N "internal state" copies of the velocity fields is a step in a wrong direction: the $N \to \infty$ limit has no physical sense and produces wrong results, whereas the treatment of the first order terms in $1/N$ is even more complicated than the initial problem of description of turbulence in the frame of the Navier-Stokes equation.

Consider an incompressible velocity field $\mathbf{u}(\mathbf{r}, t)$ which is the solution of the Navier Stokes equations

$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \nu \nabla^2 \mathbf{u} + \boldsymbol{\nabla} p = 0 \ , \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \ , \quad (1)$$

where ν is the kinematic viscosity and p is the pressure. We assume that there exist appropriate boundary conditions to maintain a high Reynolds number flow. It has been suggested in the literature, and especially recently [1], that it is advantageous to consider a generalization of this equation to a situation in which there are N copies of the velocity field labeled by an "internal" index s:

$$\mathbf{u}(\mathbf{r}, t) \Rightarrow \mathbf{u}_s(\mathbf{r}, t), \quad s = 1, 2, \dots, N \ . \quad (2)$$

In terms of these copies one writes the generalized equation for incompressible fields $\mathbf{u}_s$:

$$\partial \mathbf{u}_s/\partial t + A_{slm}(\mathbf{u}_l \cdot \boldsymbol{\nabla})\mathbf{u}_m - \nu \nabla^2 \mathbf{u}_s + \boldsymbol{\nabla} p_s = 0 \ . \quad (3)$$

The hope is that the statistical properties of this theory are simpler to elucidate than those of (1) in the limit $N \to \infty$. We will show here that this hope is not realized when the proper symmetries are taken into account.

Consider first the implication of Galilean invariance. Our theory has to remain invariant to the transformation

$$\mathbf{r} \to \mathbf{r}' \equiv \mathbf{r} - \mathbf{U}_0 t \ , \quad \mathbf{u}_s \to \mathbf{u}'_s \equiv \mathbf{u}_s + h_s \cdot \mathbf{U}_0 \ . \quad (4)$$

In (4) h_s is a scalar in 3-space, but a vector in the internal state space. In case when the velocity fields $\mathbf{u}_s$ have a physical sense (like in the two-fluid models of plasma) Galilean invariance requires $h_s = 1$ for every s. However our future conclusions are independent of the particular choice of h_s, which may be considered as a free parameter.

Applying the transformation (4), to (3) we find two additional terms. The time derivative leads to an extra term which $-(\mathbf{U}_0 \cdot \nabla)\mathbf{u}_s$, whereas the nonlinear term leads to the extra term $A_{slm} h_l (\mathbf{U}_0 \cdot \nabla)\mathbf{u}_m)$. Since these two terms must cancel, we find the constraint on A_{slm} which follows from the fundamental symmetry of hydrodynamics – Galilean invariance:

$$A_{slm} h_l = \delta_{sm} \ . \quad (5)$$

Next remind that in the known cases of simplification of a problem in the large N limit (see e.g. [2,3]) one needs to have some continuous group of symmetry, the Lie group. Important example of the group Lie are the group of rotations in 3 dimensional space SO(3) and the group of unitary matrixes 2×2 SU(2). Elements of the Lie groups g may be parametrizied by a set of continuous parameters, like the Euler angles for the SO(3) group. Following [1] and aiming possibility of $1/N$-simplification we equip the copy space with a symmetry group Lie $\mathcal{G}$. By analogy with the theories [2,3] the index s have to be considered as the label of the basis of the representations of $\mathcal{G}$. Denote by $\hat{T}(g)$ the operator that corresponds to the the element g of the group, and the transformed field by $\tilde{\mathbf{u}}_s$. Then

$$\tilde{\mathbf{u}}_s \equiv \big(\hat{T}(g)\mathbf{u}\big)_s = T_{ss'}(g)\mathbf{u}'_s \ , \quad (6)$$

where $N \times N$ matrix $T_{ss'}(g)$ is the representation of $\hat{T}(g)$ in this basis. Now we apply $\hat{T}(g)$ to the equations of motion (3) and demand invariance. Write for that these equations for $\mathbf{u}_{s'}$ and multiply $T_{ss'}(g)$ from the left. Then

$$\frac{\partial \tilde{\mathbf{u}}_s}{\partial t} + T_{ss'}(g)A_{s'lm}(\mathbf{u}_l \cdot \boldsymbol{\nabla})\mathbf{u}_m - \nu \nabla^2 \tilde{\mathbf{u}}_s + \boldsymbol{\nabla} \tilde{p}_s = 0. \quad (7)$$

Next we use the fact that $\mathcal{G}$ is a group and therefore the matrix $T_{ss'}$ has an inverse $\mathbf{u}_s = T^{-1}_{ss'}(g)\tilde{\mathbf{u}}_{s'}$. Substituting in (7) and demanding invariance leads to the constraint

$$T_{ss'}(g)A_{s'l'm'}T^{-1}_{l'l}(g)T^{-1}_{m'm}(g) = A_{slm} \ . \quad (8)$$

This is the constraint on A_{slm} which follows from the fact that the set of Eqs. (3) is invariant with respect to the transformation (6) of the group $\mathcal{G}$.

Now we have two constraints (5) and (8) on the same tensor A_{slm} which will be considered as a restriction on the allowed form of the transformation (6) of the group $\mathcal{G}$. To find this restriction we multiply the left hand side of (8) by h_l and sum up on l. Together with (5) it gives:

$$T_{ss'}(g)A_{s'l'm'}T^{-1}_{l'l}(g)T^{-1}_{m'm}(g)h_l = \delta_{sm} \ . \quad (9)$$

This equation is simplified, by multiplying on the left by T^{-1} and on the right by T to obtain $A_{sml'}T^{-1}_{l'l}(g)h_l = \delta_{sm}$. It appears now that if we have uncountably many constraints (since g is continuous) on the finite dimensional tensor A_{sml}. The only way to remove the over determination is to demand that $T^{-1}_{l'l}(g)$ is g-independent. Because for the identical transformation ($g = e$) $T_{ss'}(e) = \delta_{ss'}$ we have

$$T_{ss'}(g) = \delta_{ss'}\ . \tag{10}$$

Finally, if we apply (**??**) to (8) we find that the equipment of the copy space with a continuous symmetry group leaves A_{slm} without any additional constraint. Equation (8) becomes an empty identity. The way to understand this startling result is that the requirement of Galilean invariance introduced an anisotropic ray h_s in the copy space, and there does not exist a nontrivial transformation that leaves it invariant. In other words the requirement of the $\mathcal{G}$-symmetry of the set of Eqs. (3) itself gives nothing because the whole problem includes the Galilean invariance (4) which contradicts to this symmetry.

The point to understand now is that the conclusion that the coefficients A_{slm} are unconstrained eliminates any hoped for advantages of the $1/N$ expansion. The technical remark is that in successful applications of this method the one-loop diagrams have an N weight that is larger than that of two or more-loop diagrams. The way this works in practice [1] is through some "coherency" conditions in A_{slm} that cause a 2-loop diagram to count N less than in a product of two 1-loop diagrams. However, if A_{slm} is unconstrained by the choice of the symmetry group, there is no loss of N factors in the 2-loop diagrams compared with the 1-loop diagrams, and no simplification of the diagrammatics appears in the $N \to \infty$ limit.

Yet in a recent contribution there was an attempt to overcome this hopeless situation by a clever trick. The idea of [1] is to consider the N-dimensional space (2) of additional unphysical vector fields $\mathbf{u}_s(\mathbf{r}, t)$ in which Galilean invariance (4) is *not respected*, coupled to one physical velocity field $\mathbf{u}_0(\mathbf{r}, t)$ for which Galilean invariance is *retained*. The result of this attempt was a prediction that the scaling exponent ζ_2 of the second order structure function in the $N \to \infty$ limit attains the value of $\zeta_{2,N\to\infty} = 1/2 = 0.5$ which differs from the experimental value $\zeta_{2,\mathrm{exp}} \simeq 0.70$ (see e.g. [4]) much large than the well known Kolmogorov 1941 prediction $\zeta_{2,\mathrm{K41}} = 2/3 \simeq 0.67$. There is an immediate reason for worry about this prediction. It has been observed many years ago by Kraichnan [5] that disregarding the Galilean symmetry and truncating the diagrammatic at a finite order produces a $1/2$ prediction for ζ_2. Is it possible that the result reported in [1] is essentially identical to that? We think so. In fact, on page 3761 of [1] one finds the following observations: "*Since we have seen that the zero mode* [the physical velocity $\mathbf{u}_0(\mathbf{r}, t)$] *contributes negligible to the internal bonds of a graph when $N \to \infty$, we may now completely neglect the zero mode in resuming these diagrams.* This means that the theory for the N unphysical modes in the $N \to \infty$ limit is decoupled from the physical velocity field and becomes a theory in which Galilean invariance has been totally discarded. Then we find "*In particular, we shall now see that the Green's function and the double correlator* [of the physical velocity $\mathbf{u}_0(\mathbf{r}, t)$] *may be expressed completely in terms of the Green's function and double correlator* [of the unphysical modes $\mathbf{u}_s(\mathbf{r}, t)$]. It follows from the possibility to neglect (for $N \to \infty$) the own nonlinearity of of the physical velocity with respect to N nonlinear contributions of unphysical modes. Thus the scaling behavior of the physical velocity is totally determined by the unphysical modes. The conclusion is that the Galilean invariance disappears from the problem in the formulation of Ref. [1] in $N \to \infty$ limit.

Clearly for $N = 0$ we have the initial Navier-Stokes based formulation of the hydrodynamic turbulence. Therefore one may hope to have something reasonable [1] in the next terms of $1/N$ expansion when some tail of the Galilean invariance will recover. On the page 3745 of [1] we found: "*A real test of our approach would be to compute the first correction in powers of* $1/N \ldots$". This corrections originate from not only two-loops diagrams of unphysical fields but also from ALL ORDERS diagrams with respect to own nonlinearity of the physical velocity field. There are no $1/N$ simplification in the later series. Therefore the task to find the $1/N$ corrections is equivalent to solution the problem of the Navier-Stokes hydrodynamic turbulence.

In summary, it was suggested the multicomponent generalization of the problem of turbulence such that the zero-order approximation is solvable. However the next step occurs to be as complicated as the whole solution of the initial problem. It is not unexpected and happens always when the zero-order problem has no relation to the initial one. In considered case this is so because the Galilean invariance (broken in zero-order) as stressed years ago by Kraichnan, is crucial for hydrodynamics. In short, the multicomponent extension of the velocity field discussed in [1] is a step in a wrong direction which makes the following steps even more complicated.

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